Optimal control of martingales in a radially symmetric environment

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29^{th} April 2021

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Joint work with Alexander Cox (University of Bath)

Problem statement

Minimise

$$\mathbb{E}\left[\int_0^{\tau_D} f(X_s)\,\mathrm{d}s + g(X_{\tau_D})\right]$$

over continuous martingales X with fixed quadratic variation $\langle X\rangle_t = t$, defined on some bounded domain

$$D \subset \mathbb{R}^d, \quad d \geq 2.$$

Motivation

An example of a problem in mathematical finance is to find

$$v(\mathbb{P}) = \sup_{\tau} \mathbb{E}\left[\int_0^{\tau} f(X_s)\,\mathrm{d}s + g(X_\tau)\right].$$

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Robust finance is concerned with finding model-independent bounds such as

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We study the inner optimisation problem and are interested in the structure of multidimensional martingales.

Problem formulation

Optimal behaviour

Construction of explicit solution

Extension of results

An SDE with no strong solution

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Let $\mathcal{U}$ be the set of U-valued $\mathbb{F}$-progressively measurable processes.

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Dynamics: For $x \in D$ and $\nu \in \mathcal{U}$, define X^ν by the stochastic integral

$$X_t^\nu = x + \int_0^t \nu_s \, \mathrm{d}B_s, \quad t \geq 0.$$

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Value function: Find the value function $v \colon D \to \mathbb{R}$,

$$v(x) := \inf_{\nu \in \mathcal{U}} \mathbb{E}^x \left[\int_0^\tau f(X_s^\nu) \, \mathrm{d}s + g(X_\tau^\nu) \right]$$

Markov controls

Example: Let $\sigma : D \to U$ be Lipschitz. Then there is a unique strong solution X^σ of the SDE

$$\mathrm{d}X_t = \sigma(X_t)\,\mathrm{d}B_t, \quad X_0 = x.$$

Define $\nu_t = \sigma(X_t^\sigma)$ for all $t \geq 0$.

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Define $\nu_t = \sigma(X_t^\sigma)$ for all $t \geq 0$.Then $\nu \in \mathcal{U}$ and

$$X_t^\sigma = x + \int_0^t \nu_s\,\mathrm{d}B_s = X_t^\nu.$$

This ν is an example of a Markov control.

Assumptions

$$v(x) := \inf_{\nu \in \mathcal{U}} \mathbb{E}^x \left[\int_0^\tau f(X_s^\nu) \, \mathrm{d}s + g(X_\tau^\nu) \right],$$

1. $D = B_R(0) \subset \mathbb{R}^d$
2. f is radially symmetric; i.e. $f(x) = \tilde{f}(|x|)$
3. g is constant

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5. $\tilde{f}'_+(r)$ exists for all $r \geq 0$ and changes sign finitely many times
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Optimal behaviour

Radial motion

Optimal behaviour for $\tilde{f}$ monotonically increasing

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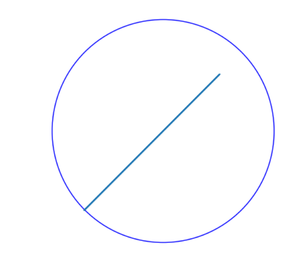
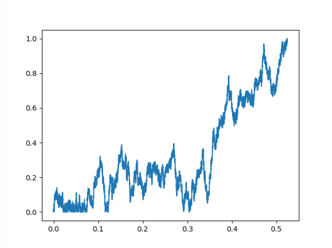


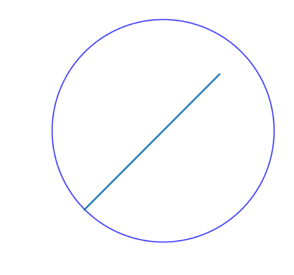
Sample path of X_t



Sample path of R_t

Radial motion

Optimal behaviour for $\tilde{f}$ monotonically increasing



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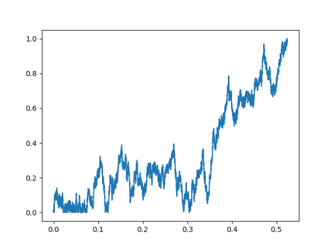

Sample path of R_t

- Control: $$\nu_t \equiv \frac{1}{|x|}[x; 0; \ldots ; 0]$$
- Radius process: $$\mathrm{d}R_t = \mathrm{d}W_t$$

Tangential motion

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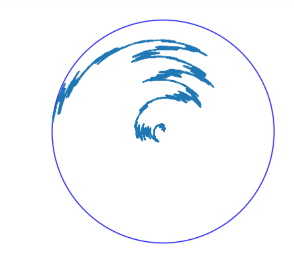

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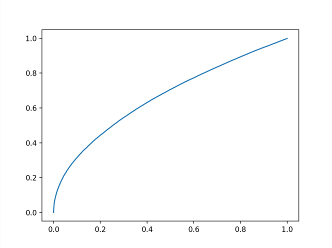

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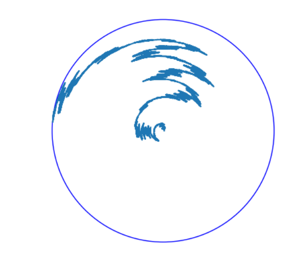

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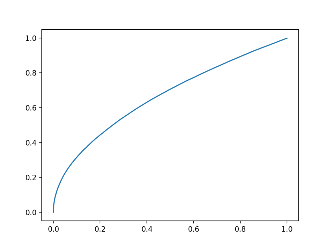
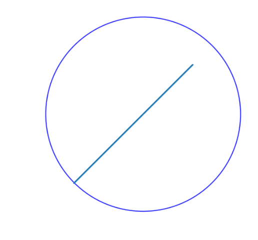


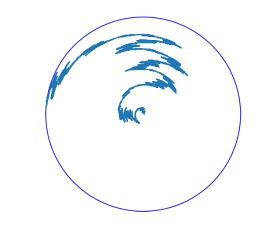
Sample path of R_t

- Control: $$\nu_t = \sigma(X_t) = \frac{1}{|X_t|}[X_t^{\perp}; 0; \ldots ; 0]$$
- Radius process: $$\mathrm{d}R_t = \frac{1}{2R_t}\,\mathrm{d}t$$ $\Rightarrow R_t = \sqrt{|x| + t}$

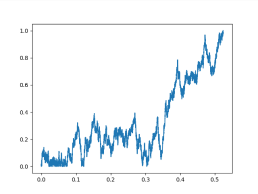
Two optimal behaviour regimes



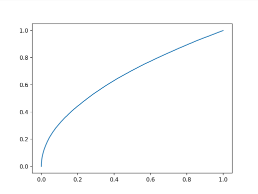
(a) Sample path of radial motion



(b) Sample path of tangential motion



(c) Sample path of radius process for (a)



(d) Sample path of radius process for (b)

Two optimal behaviour regimes

Claim

For any f satisfying Assumptions 1–6, an optimal strategy is to switch between radial and tangential motion.

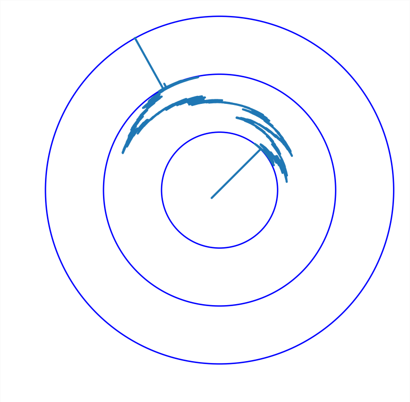


Figure 4: A possible optimal trajectory

Construction of explicit solution

Method of solution

1. Prove that the value function v is the unique viscosity solution of

$$\begin{cases} -\frac{1}{2}\inf_{\sigma\in U}\operatorname{Tr}(D^2 v\sigma\sigma^\top) = f & \text{in} \quad D \\ v = g & \text{on} \quad \partial D \end{cases} \tag{HJB}$$

2. Find switching points to construct candidate value function V
3. Show that the candidate function V solves (HJB)

Construction of value function

Claim that the optimal strategy is to switch between radial and tangential motion.

Then $v(x) = \tilde{v}(|x|)$, where

- $\tilde{v}(R) = g$
- and, for $r \in (0, R)$, either

$$dR_t = dW_t \Rightarrow \quad -\frac{1}{2}\tilde{v}''(r) = \tilde{f}(r), \quad \text{or}$$

$$dR_t = \frac{1}{2R_t}dt \Rightarrow \quad -\frac{1}{2r}\tilde{v}'(r) = \tilde{f}(r).$$

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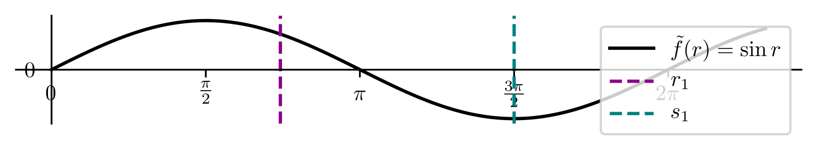
To minimise

$$\tilde{v}(r) = g - \int_r^R \tilde{v}'(s)\,\mathrm{d}s,$$

we seek to maximise $\tilde{v}'(r)$.

An example

Consider the cost function $f(x) = \sin(|x|)$



Switching points

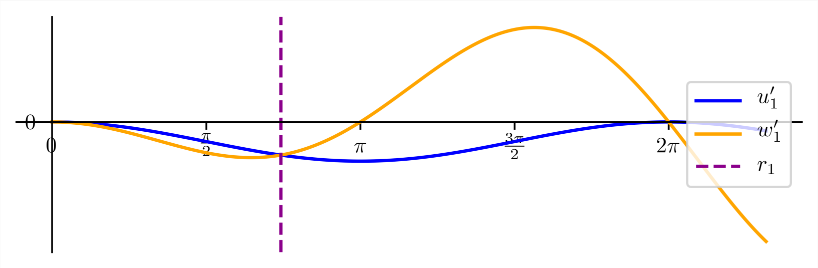
$\tilde{v}''(r) = -2\tilde{f}(r)$ $\quad$ $\tilde{v}'(r) = -2r\tilde{f}(r)$ $\quad$ $\tilde{v}''(r) = -2\tilde{f}(r)$

0 $\quad$ r_1 $\quad$ 1st order $\quad$ s_1

Return to the example

$$\color{blue}{u_1}''(r) = -2\tilde{f}(r), \quad (\color{blue}{u_1})'_+(0) = 0$$

$$\color{orange}{w_1}'(r) = -2r\tilde{f}(r)$$



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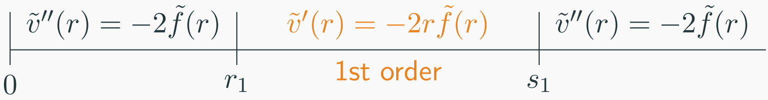
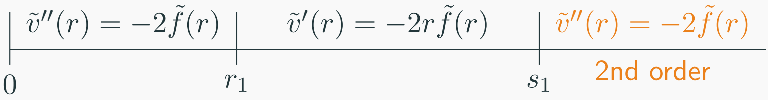


Switching point is determined by

$$r_1 = \inf\{r > s_0 \colon \int_0^r \tilde{f}(s)\,\mathrm{d}s > r\tilde{f}(r)\}.$$

By continuity of f, we have smooth fit at r_1.

Switching points

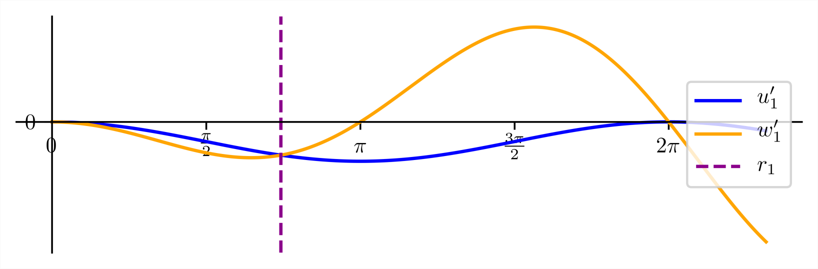


We need to enforce smooth fit at s_1

Return to the example

$$w_1'(r) = -2r\tilde{f}(r)$$

$$u_2''(r) = -2\tilde{f}(r), \quad (u_2)'_+(s_1) = w_1'(s_1)$$



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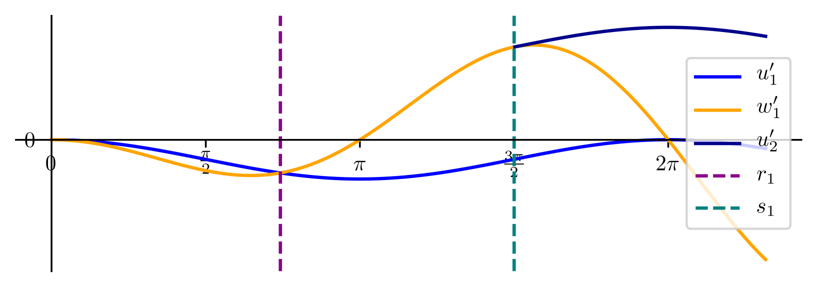

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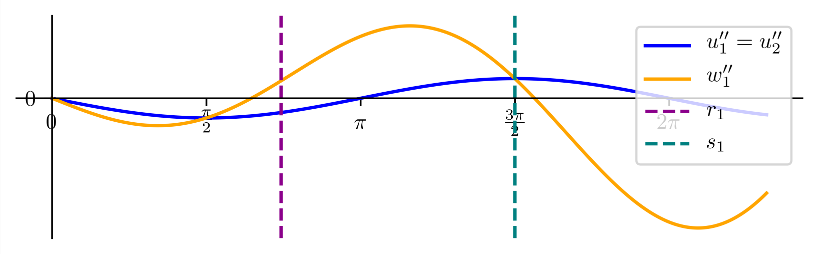

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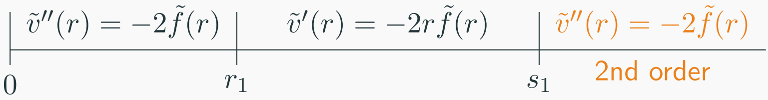
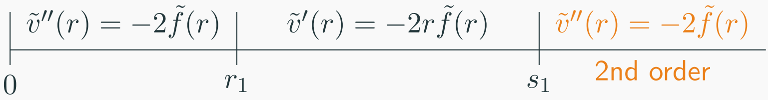


We need to enforce smooth fit at s_1, and we need a 2nd order condition to determine the switching point:

$$s_1 = \inf\{r > s_0 \colon \tilde{v}''_+(r) < -2\tilde{f}(r)\}.$$

Switching points



We need to enforce smooth fit at s_1, and we need a 2nd order condition to determine the switching point:

$$s_1 = \inf\{r > s_0 \colon \tilde{f}'_+(r) > 0\}.$$

Switching points

Continue in this way to construct a sequence of switching points

$$s_0 < r_1 < \ldots < r_i < s_i < \ldots,$$

with

$$r_i := \inf \left\{ r > s_{i-1} \colon \int_{s_{i-1}}^{r} \tilde{f}(s)\,\mathrm{d}s > r\tilde{f}(r) \right\},$$

and

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We arrive at the following candidate value function $V : D \to \mathbb{R}$.

Candidate value function

Case 1: If $\tilde{f}$ is increasing in $(0,\eta)$, then set $s_0 = 0$ and let $K \in \mathbb{N}$ be such that $R \in (s_{K-1}, s_K]$. For $x \in D$, define

$$V(x) = g - 2\int_{R\vee r_K}^{s_K} s\tilde{f}(s)\,\mathrm{d}s \\ - 2(r_K - R\wedge r_K)s_{K-1}\tilde{f}(s_{K-1}) - 2\int_{R\wedge r_K}^{r_K}\int_{s_{K-1}}^{s}\tilde{f}(t)\,\mathrm{d}t\,\mathrm{d}s \\ + 2\sum_{i=1}^{K}\mathbb{1}_{\{(s_{i-1},s_i]\}}(|x|)\Big[(r_i - |x|\wedge r_i)s_{i-1}\tilde{f}(s_{-1}) \\ + \int_{|x|\wedge r_i}^{r_i}\int_{s_{i-1}}^{s}\tilde{f}(t)\,\mathrm{d}t\,\mathrm{d}s + \int_{|x|\vee r_i}^{s_i} s\tilde{f}(s)\,\mathrm{d}s + \mathfrak{F}_i^K\Big].$$

Candidate value function

Case 2: If $\tilde{f}$ is decreasing in $(0, \eta)$, then set $r_0 = 0$ and let $L \in \mathbb{N}$ be such that $R \in (r_L, r_{L+1}]$. For $x \in D$, define

$$\begin{aligned} V(x) &= g - 2\int_{R \wedge s_L}^{s_L} s\tilde{f}(s)\,\mathrm{d}s \\ &\quad + 2(R \vee s_L - s_L)s_L\tilde{f}(s_L) + 2\int_{s_L}^{R \vee s_L}\int_{s_L}^{s}\tilde{f}(t)\,\mathrm{d}t\,\mathrm{d}s \\ &\quad + 2\sum_{i=0}^{L}\mathbb{1}_{\{(r_i, r_{i+1}]\}}(|x|)\left[\int_{|x| \wedge s_i}^{s_i} s\tilde{f}(s)\,\mathrm{d}s - (|x| \vee s_i - s_i)s_i\tilde{f}(s_i)\right. \\ &\qquad\qquad \left. - \int_{s_i}^{|x| \vee s_i}\int_{s_i}^{s}\tilde{f}(t)\,\mathrm{d}t\,\mathrm{d}s + \mathfrak{F}_i^L\right]. \end{aligned}$$

Candidate value function

There exist constants C_i, $\tilde{C}_i$ such that

$$V(x) =$$

$$\begin{cases} -2\int_{s_{i-1}}^{|x|}\int_{s_{i-1}}^{s}\tilde{f}(t)\,\mathrm{d}t\,\mathrm{d}s - 2\,|x|\,s_{i-1}\tilde{f}(s_{i-1}) + C_i, & |x| \in [s_{i-1}, r_i], \\ -2\int_{r_i}^{|x|} s\tilde{f}(s)\,\mathrm{d}s + \tilde{C}_i, & |x| \in [r_i, s_i]. \end{cases}$$

Theorem [Cox and R. 2021+]

Under Assumptions 1–6, the value function is given by

$$v = V.$$

Idea of the proof

1. Prove that the value function v
   - is continuous and semi-convex
   - satisfies a dynamic programming principle
   - is the unique viscosity solution of

$$\begin{cases} -\frac{1}{2}\inf_{\sigma \in U} \operatorname{Tr}(D^2 v \sigma \sigma^\top) = f & \text{in} \quad D \\ v = g & \text{on} \quad \partial D \end{cases} \tag{HJB}$$

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2. Verify that V solves (HJB)
3. Conclude that $v = V$

Extension of results

Original assumptions

We now relax the assumptions:

$$v(x) := \inf_{\mathbb{P} \in \mathcal{P}_x} \mathbb{E}^{\mathbb{P}} \left[\int_0^\tau f(X_s)\, \mathrm{d}s + g(X_\tau) \right],$$

1. $D = B_R(0)$
2. f radially symmetric; i.e. $f(x) = \tilde{f}(|x|)$
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Relaxed assumptions

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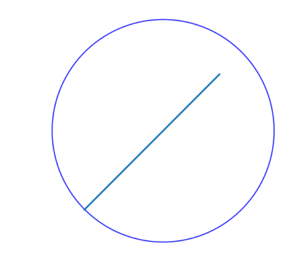
2. f radially symmetric; i.e. $f(x) = \tilde{f}(|x|)$
3. g constant
4. f continuous in $D \setminus \{0\}$
5. $\tilde{f}'_+(r)$ exists for all $r \geq 0$ and changes sign finitely many times
6. $\tilde{f}$ is monotone near the origin

Optimal behaviour



Radial motion

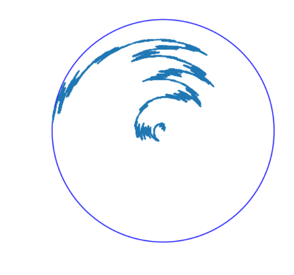
$\nu_t \equiv \frac{1}{|x|}[x; 0; \dots ; 0]$

$x \neq 0$

If $x = 0$,

$\nu_t = e_1$

$dX_t = e_1 \, dB_t$



Tangential motion

$\nu_t = \sigma(X_t) = \frac{1}{|X_t|}[X_t^{\perp}; 0; \dots ; 0]$

$x \neq 0$

$dX_t = \sigma(X_t) \, dB_t$

$x = 0$

?

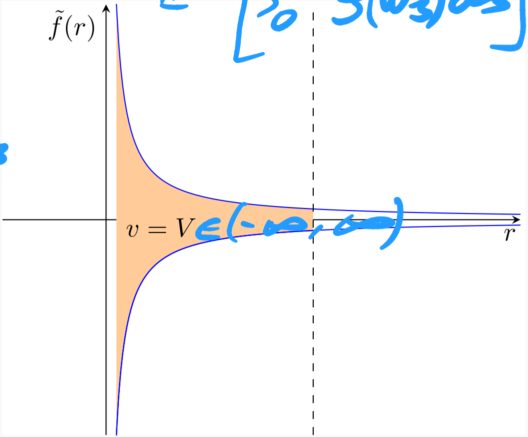
Extension of results

Suppose that

$$\int_0^r \tilde{f}(s)\,\mathrm{d}s \in (-\infty, \infty).$$

$$\mathbb{E}^0\left[\int_0^{\tau_\delta} \tilde{f}(W_s)\,ds\right] \sim \delta \int_0^\delta \tilde{f}(s)\,ds \to 0$$

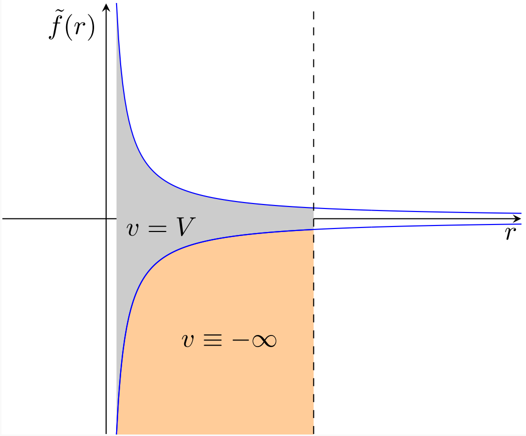
E.g. $\tilde{f}(r) \sim -\frac{1}{r^\beta}$, $\beta < 1$



Extension of results

Suppose that

$$\int_0^r \tilde{f}(s)\,\mathrm{d}s = -\infty.$$



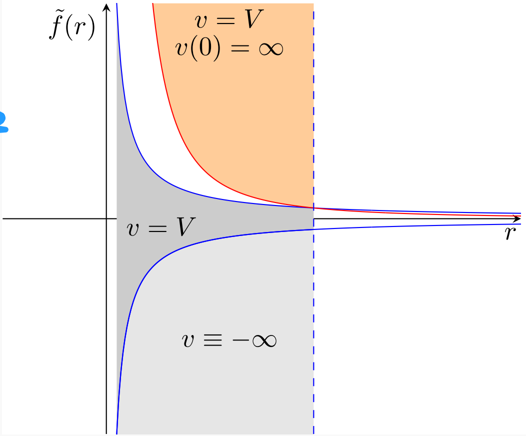
Extension of results

Suppose that

$$\int_0^r s\tilde{f}(s)\,\mathrm{d}s = +\infty.$$

E.g.

$\tilde{f}(r) \sim \frac{1}{r^2}$



Extension of results

Suppose that

$$\int_0^r \tilde{f}(s) = \infty \quad \text{and} \quad \int_0^r s\tilde{f}(s)\,\mathrm{d}s < \infty.$$

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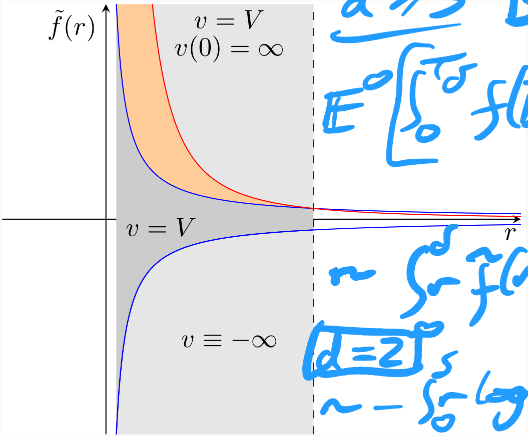

$d \geq 3$ B d-dim.

$E^0\left[\int_0^{\tau_\sigma} f(B_s)\,ds\right]$

$\sim \int_0^\sigma \tilde{f}(r)\,dr$

$d=2$

$\sim -\int_0^\sigma \log r\,\tilde{f}(dr) = \infty$

An SDE with no strong solution

An SDE with no strong solution

Fix $d = 2$ and let B be a one-dimensional Brownian motion.

Theorem [Larsson and Ruf, 2020]

The SDE

$$\mathrm{d}X_t = \frac{1}{|X_t|} X_t^{\perp} \, \mathrm{d}B_t; \quad X_0 = 0 \tag{1}$$

has a weak solution.

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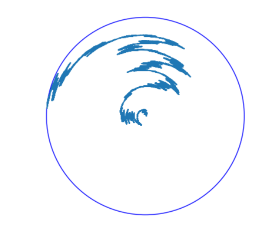
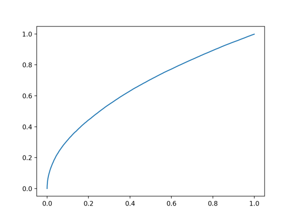


Sample path of X_t



Sample path of R_t

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Fix $d = 2$ and let B be a one-dimensional Brownian motion.

Theorem [Larsson and Ruf, 2020]

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has a weak solution.

Theorem [Cox and R. 2021+]

The SDE (1) has no strong solution.

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Fix $d = 2$ and let B be a one-dimensional Brownian motion.

Theorem [Larsson and Ruf, 2020]

The SDE

$$\mathrm{d}X_t = \frac{1}{|X_t|} X_t^{\perp}\,\mathrm{d}B_t; \quad X_0 = 0 \tag{1}$$

has a weak solution.

$$dX_t = b(t, X_\cdot)\,dt + dB_t$$

Theorem [Cox and R. 2021+]

The SDE (1) has no strong solution.

- Proof uses ideas from the study of Tsirelson's equation.
- We use properties of circular Brownian motion, as proved in [Émery and Schachermayer, 1999].

An SDE with no strong solution

Theorem [Cox and R. 2020+]

Fix $d = 2$ and suppose that

$$\int_0^r \tilde{f}(s)\,\mathrm{d}s = \infty \quad \text{and} \quad \int_0^r s\tilde{f}(s)\,\mathrm{d}s < \infty.$$

Then $v = V < \infty$.

An SDE with no strong solution

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Then $v = V < \infty$.

- A weak solution X of (1) generates a Brownian filtration by [Émery and Schachermayer, 1999] $\mathcal{F}^X = \mathcal{F}^W$

An SDE with no strong solution

$\nu_t = \sigma(X_t)$

$dX_t = \sigma(X_t)\, dB_t$

Theorem [Cox and R. 2020+]
Fix $d = 2$ and suppose that

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- A weak solution X of (1) generates a Brownian filtration by [Émery and Schachermayer, 1999]
- There exists $\nu^\star \in \mathcal{U}$ such that

$$Y_t := \int_0^t \nu_s^\star \,\mathrm{d}B_s.$$

satisfies $Y \overset{\text{law}}{=} X$.

An SDE with no strong solution

Theorem [Cox and R. 2020+]

Fix $d = 2$ and suppose that

$$\int_0^r \tilde{f}(s)\,\mathrm{d}s = \infty \quad \text{and} \quad \int_0^r s\tilde{f}(s)\,\mathrm{d}s < \infty.$$

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Conjecture

Fix $d = 2$ and suppose that

$$\int_0^r \tilde{f}(s)\,\mathrm{d}s = \infty \quad \text{and} \quad \int_0^r s\tilde{f}(s)\,\mathrm{d}s < \infty.$$

Then $v(0) < v^M(0)$.

Summary

- Constructed the value function explicitly for continuous radially symmetric costs

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- Extended this result to costs that become infinite at the origin
- Conjecture that there exists a case where the value at the origin is greater when restricted to Markov controls:
  - Proved that an SDE describing tangential motion has a weak solution but no strong solution started from the origin
  - Proved that an approximating sequence of SDEs have no strong solution
  - Require to prove that these SDEs have no strong solution when driven by 2D Brownian motion

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Relative Arbitrage: Sharp Time Horizons and Motion by Curvature.
arXiv:2003.13601 [math, q-fin].

Summary

- Constructed the value function explicitly for continuous radially symmetric costs
- Extended this result to costs that become infinite at the origin
- Conjecture that there exists a case where the value at the origin is greater when restricted to Markov controls

For details, see the thesis
Stochastic control problems for multidimensional martingales

https://mat.univie.ac.at/∼brobinson