Bicausal optimal transport for SDEs with irregular coefficients

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Goal

Scalar time-homogeneous SDEs, $b, \bar{b} \colon \mathbb{R} \to \mathbb{R}$, $\sigma, \bar{\sigma} \colon \mathbb{R} \to [0, \infty)$,

$$\begin{aligned} dX_t &= b(X_t)dt + \sigma(X_t)dW_t & \rightsquigarrow \quad & \mu = \mathrm{Law}\left((X_t)_{t\in[0,T]}\right) \\ d\bar{X}_t &= \bar{b}(\bar{X}_t)dt + \bar{\sigma}(\bar{X}_t)d\bar{W}_t & \rightsquigarrow \quad & \nu = \mathrm{Law}\left((\bar{X}_t)_{t\in[0,T]}\right) \end{aligned}$$

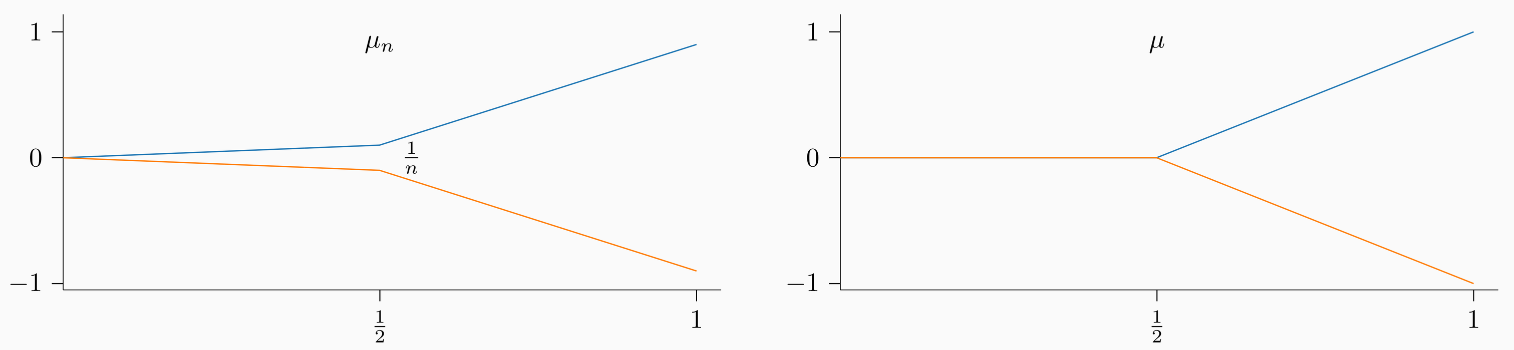
Compute a suitable distance between μ and ν.

Adapted Wasserstein

Canonical distance between probability measures is the Wasserstein distance

$$\mathcal{W}_p^p(\mu, \nu) = \inf_{\pi \in \mathrm{Cpl}(\mu,\nu)} \int_\Omega \int_0^T |\omega_s - \bar{\omega}_s|^p \, dt \, d\pi(\omega, \bar{\omega})$$

Example:



- Optimal coupling is to match the blue paths and the orange paths, and so $\mathcal{W}_p(\mu_n, \mu) \to 0$
- A sequence of deterministic processes converging to a martingale
- This is not a good distance between stochastic processes

Define the adapted Wasserstein distance $\mathcal{AW}_p(\mu, \nu)$ by replacing $\mathrm{Cpl}(\mu, \nu)$ with

$$\mathrm{Cpl}_{\mathrm{bc}}(\mu, \nu) = \{\pi \in \mathrm{Cpl}(\mu, \nu) \colon \ \pi \text{ is bicausal}\}$$

- $\pi = \mathrm{Law}(X, \bar{X})$ is bicausal if *given the past of* X*, the future of* $\bar{X}$ *is independent of* X, and vice-versa
- The optimal coupling above is no longer admissible
- $\mathcal{AW}_2(\mu_n, \mu) \approx \frac{1}{2}$ for all n large

Assumptions

(1) b **has at most exponential growth and finitely many points of discontinuity** ξ_k**,** σ **is Lipschitz and** $\sigma(\xi_k) \neq 0$**.**

(2) b is bounded and measurable, σ is bounded, α-Hölder ($\alpha \in [1/2, 1]$), and uniformly non-degenerate.

(3) b, σ are continuous with linear growth, and pathwise uniqueness holds for the SDE.

Main Theorem

Suppose that (b, σ) and $(\bar{b}, \bar{\sigma})$ each satisfy **one of** Assumption 1 **or** Assumption 2 **or** Assumption 3.

Then the synchronous coupling π^{sync} attains $\mathcal{AW}_p(\mu, \nu)$, where $\pi^{\mathrm{sync}} = \mathrm{Law}(X, \bar{X})$ with $W = \bar{W}$.

Numerical Scheme

Transformation: For (b, σ) satisfying **Assumption 1**, define $Z = G(X)$, so that for some continuous $\tilde{b}, \tilde{\sigma}$,

$$dZ_t = \tilde{b}(Z_t)dt + \tilde{\sigma}(Z_t)dW_t.$$

Transformed semi-implicit Euler scheme: Let $N \in \mathbb{N}$, $h = T/N$,

$$Z_0^h = Z_0, \quad Z_{(k+1)h}^h = Z_{kh}^h + h\tilde{b}(Z_{(k+1)h}^h) + \tilde{\sigma}(Z_{kh}^h)\Delta W_{k+1}^h,$$

with truncated Brownian increments ΔW_{k+1}^h.

Define X^h by $X_{kh}^h = G^{-1}(Z_{kh}^h)$, for $k \in \{0, \dots, N\}$.

Convergence Result

Let (b, σ) satisfy **Assumption 1**. Then the SDE has a unique strong solution and, for all $p \geq 1$, there exists $C_p \geq 0$ such that

$$\mathbb{E}\left[|X_T - X_T^h|^p\right]^{\frac{1}{p}} \leq \begin{cases} C_p h^{\frac{1}{3}}, & p \in [1, 2], \\ C_p h^{\frac{1}{p(p-1)}}, & p \geq 2. \end{cases}$$

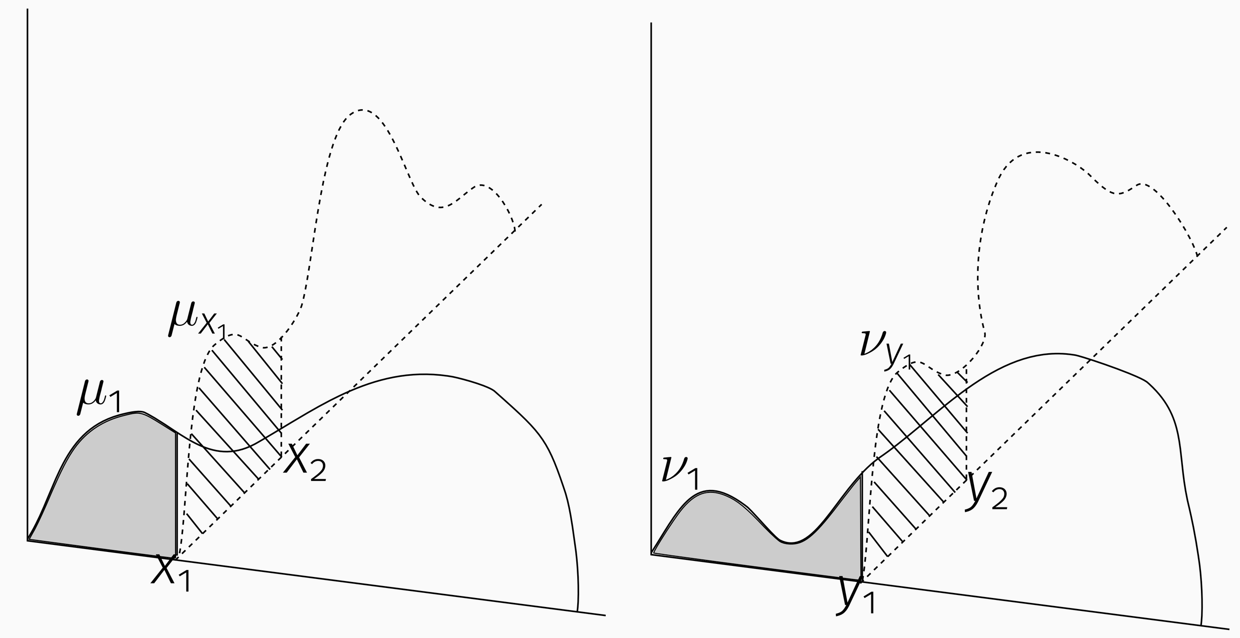
Knothe–Rosenblatt

For μ^h, ν^h on $\mathbb{R}^N$, $U_1, \dots, U_N \overset{iid}{\sim} \mathcal{U}[0, 1]$, $X_1 = F^{-1}_{\mu_1^h}(U_1)$, $Y_1 = F^{-1}_{\nu_1^h}(U_1)$,

$$X_k = F^{-1}_{\mu^h_{X_1,\dots,X_{k-1}}}(U_k), \qquad Y_k = F^{-1}_{\nu^h_{Y_1,\dots,Y_{k-1}}}(U_k)$$

The Knothe–Rosenblatt rearrangement between μ^h, ν^h is

$$\pi^{\mathrm{KR}} = \mathrm{Law}(X_1, \dots, X_n, Y_1, \dots, Y_N)$$



Knothe–Rosenblatt rearrangement for $N = 2$

Proof Idea

- Under **Assumption 1**, discretise $X, \bar{X}$ by $X^h, \bar{X}^h$.
- $\mu^h = \mathrm{Law}(X^h)$ and $\nu^h = \mathrm{Law}(\bar{X}^h)$ are stochastically increasing.
- π^{KR} attains $\mathcal{AW}_p(\mu^h, \nu^h)$ [Rüschendorf '85].
- $\mathcal{AW}_p(\mu^h, \nu^h) \xrightarrow{h \to 0} \mathcal{AW}_p(\mu, \nu)$ and π^{sync} attains $\mathcal{AW}_p(\mu, \nu)$.

References

[1] Julio Backhoff-Veraguas, Sigrid Källblad, and Benjamin A Robinson. Adapted Wasserstein distance between the laws of SDEs. *arXiv:2209.03243*, 2022.

[2] Benjamin A. Robinson and Michaela Szölgyenyi. Bicausal optimal transport for SDEs with irregular coefficients. *Preprint*, 2024.

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