Bicausal optimal transport for SDEs with irregular coefficients

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Joint work with

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Universität Klagenfurt	University of Vienna	KTH Stockholm

Adapted Wasserstein distance between the laws of SDEs
(with J. Backhoff-Veraguas and S. Källblad) — http://arxiv.org/abs/2209.03243

Bicausal optimal transport for SDEs with irregular coefficients
(with M. Szölgyenyi) — https://arxiv.org/abs/2403.09941

Comparing stochastic models

Aim: Compute a measure of model uncertainty

E.g.

$$\mathbb{P} \mapsto v(\mathbb{P}) = \sup_{\alpha \in \mathcal{A}} \mathbb{E}^{\mathbb{P}}[\mathcal{J}(\omega, \alpha)]$$

Want:

- Appropriate topology on laws of stochastic processes
- Distance we can actually compute

SDEs:

- Good computational methods available
- Rich class of models, beyond Lipschitz coefficients

Main result

$b, \bar{b} \colon \mathbb{R} \to \mathbb{R}$, $\sigma, \bar{\sigma} \colon \mathbb{R} \to [0, \infty)$, $X_0 = \bar{X}_0 = x \in \mathbb{R}$,

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t$$
$$\mathrm{d}\bar{X}_t = \bar{b}(\bar{X}_t)\mathrm{d}t + \bar{\sigma}(\bar{X}_t)\mathrm{d}W_t$$

$\mu = \mathrm{Law}(X)$, $\nu = \mathrm{Law}(\bar{X})$

Theorem [R., Szölgyenyi '24+]

Under "weak assumptions" on the coefficients, we can compute an "appropriate distance" by

$$d(\mu, \nu)^2 = \mathbb{E}\left[\int_0^T |X_t - \bar{X}_t|^2 \mathrm{d}t\right], \quad \text{with } B = W.$$

Auxiliary result

$b\colon \mathbb{R} \to \mathbb{R}$, $\sigma\colon \mathbb{R} \to [0, \infty)$, $X_0 = x \in \mathbb{R}$,

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t. \qquad \text{(SDE)}$$

Assumption (A)

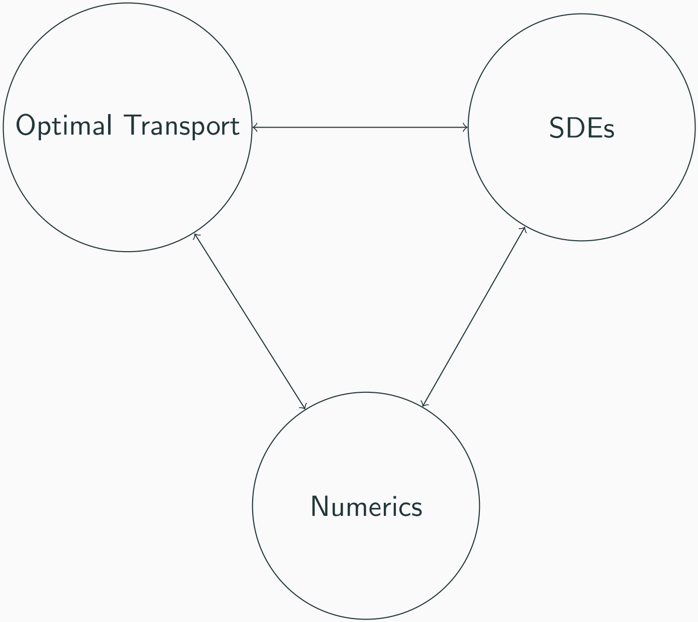
b satisfies piecewise regularity conditions and exponential growth condition,

σ is Lipschitz and non-zero at the discontinuity points of b.

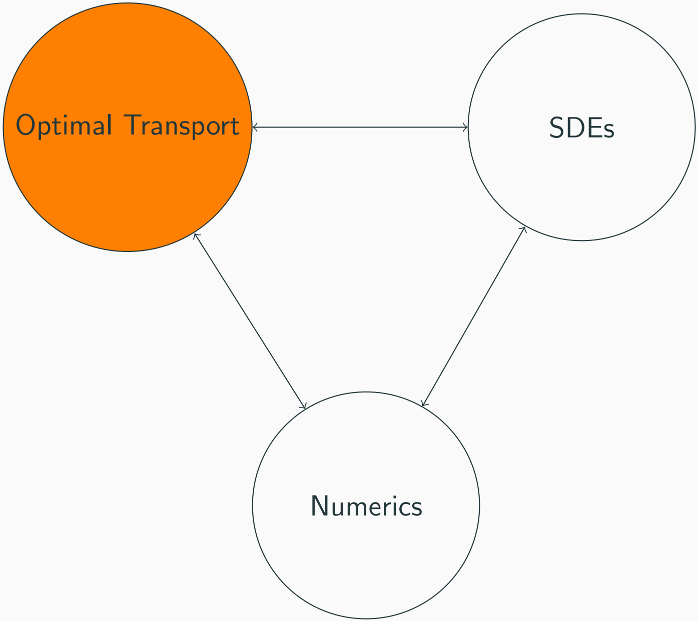
Theorem [R., Szölgyenyi '24+]

Strong existence, pathwise uniqueness, and moment bounds hold for (SDE) with coefficients satisfying (A). Moreover, for a transformation-based semi-implicit Euler scheme, we obtain strong convergence rates.

Ingredients



Ingredients



Optimal transport

Probability measures μ, ν on $\mathbb{R}^N$

Find

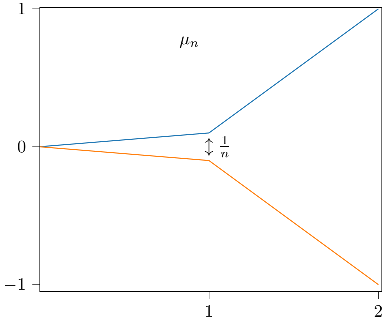
$$\mathcal{W}_2^2(\mu,\nu) := \inf_{T\colon T_\# \mu = \nu} \mathbb{E}\left[\sum_{n=1}^{N} |T_n(X) - X_n|^2\right].$$

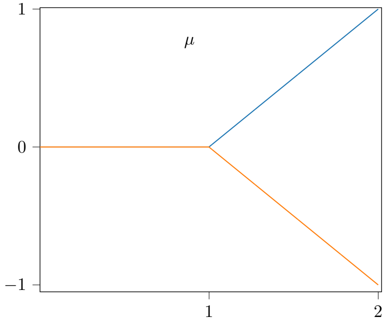
$$T(X) = (T_1(X_1, \dots, X_N), \dots, T_N(X_1, \dots, X_N))$$

Monge (1781)
Kantorovich (1942) $\rightsquigarrow T$ random: replace $(X, T(X))$ with coupling (X, Y), $X \sim \mu$, $Y \sim \nu$
Wasserstein distance metrises usual weak topology

Example

[Backhoff-Veraguas, Bartl, Beiglböck, Eder '20], [Aldous '81]





"Can get rich" "Cannot get rich"

$$V_n := \sup_\tau \mathbb{E}^{\mu_n}[X_\tau] \approx \frac{1}{2} \qquad V := \sup_\tau \mathbb{E}^{\mu}[X_\tau] = 0$$

$V_n \not\to V$ but $\mu_n \to \mu$

Adapted topology

$$\mu, \nu \in \mathcal{P}(\mathbb{R}^N) \quad \rightsquigarrow \quad \mathcal{W}_2^2(\mu, \nu) := \inf_{T:\, T_\# \mu = \nu} \mathbb{E}\left[\sum_{n=1}^{N} |T_n(X) - X_n|^2\right].$$

$$T(X) = (T_1(X_1, \ldots, X_N), \ldots, T_N(X_1, \ldots, X_N))$$

Adapted topology

$$\mu, \nu \in \mathcal{P}(\mathbb{R}^N) \quad \rightsquigarrow \quad \inf_{\substack{T:\, T_\# \mu = \nu \\ \textit{adapted}}} \mathbb{E}\left[\sum_{n=1}^{N} |T_n(X) - X_n|^2 \right].$$

$$T(X) = (T_1(X_1), T_2(X_1, X_2), \ldots, T_N(X_1, \ldots, X_N))$$

Adapted topology

$$\mu, \nu \in \mathcal{P}(\mathbb{R}^N) \quad \rightsquigarrow \quad \mathcal{AW}_2^2(\mu,\nu) := \inf_{\substack{T:\, T_\# \mu = \nu \\ \text{biadapted}}} \mathbb{E}\left[\sum_{n=1}^N |T_n(X) - X_n|^2\right].$$

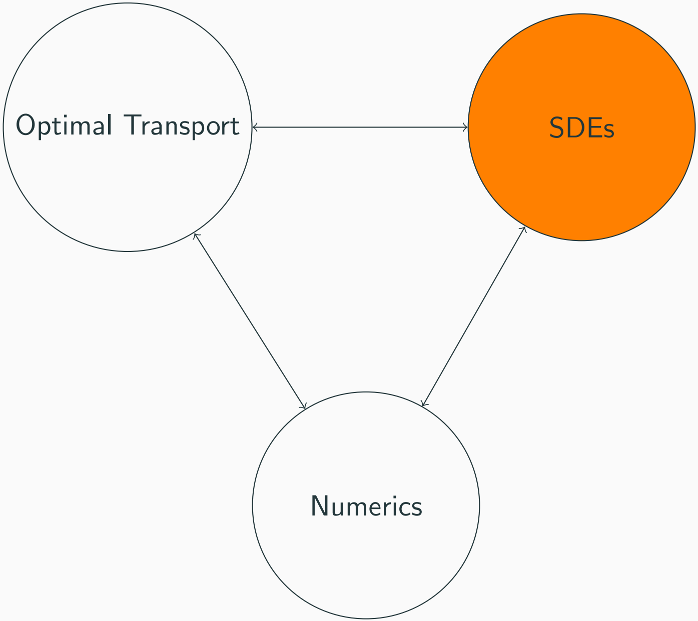
$$T(X) = (T_1(X_1), T_2(X_1, X_2), \ldots, T_N(X_1, \ldots, X_N))$$

Similar definition of adapted Wasserstein distance in continuous time w.r.t. L^p norm on $C([0,T],\mathbb{R})$

More general cost functions $\rightsquigarrow$ bicausal optimal transport

[Aldous '81][Lasalle '18][Bion-Nadal, Talay '19]
Acciaio, Backhoff-Veraguas, Bartl, Beiglböck, Eder, Hellwig, Källblad, Pammer, Pflug, Pichler, Zalaschko, ...

Ingredients



Main result

$b, \bar{b} \colon \mathbb{R} \to \mathbb{R}$, $\sigma, \bar{\sigma} \colon \mathbb{R} \to [0, \infty)$, $X_0 = \bar{X}_0 = x \in \mathbb{R}$,

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t$$
$$\mathrm{d}\bar{X}_t = \bar{b}(\bar{X}_t)\mathrm{d}t + \bar{\sigma}(\bar{X}_t)\mathrm{d}W_t$$

$\mu = \mathrm{Law}(X)$, $\nu = \mathrm{Law}(\bar{X})$

Theorem [R., Szölgyenyi '24+]

Under "weak assumptions" on the coefficients, we can compute the adapted Wasserstein distance by

$$\mathcal{AW}_2^2(\mu, \nu) = \mathbb{E}\Big[\int_0^T |X_t - \bar{X}_t|^2 \mathrm{d}t\Big], \quad \text{with } B = W.$$

Coupling SDEs

$b, \bar{b} \colon \mathbb{R} \to \mathbb{R}$, $\sigma, \bar{\sigma} \colon \mathbb{R} \to [0, \infty)$, $X_0 = \bar{X}_0 = x \in \mathbb{R}$,

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$\mu = \mathrm{Law}(X)$, $\nu = \mathrm{Law}(\bar{X})$

Theorem 1 [Backhoff-Veraguas, Källblad, R. '22]

Optimising over adapted maps T

$\Leftrightarrow$

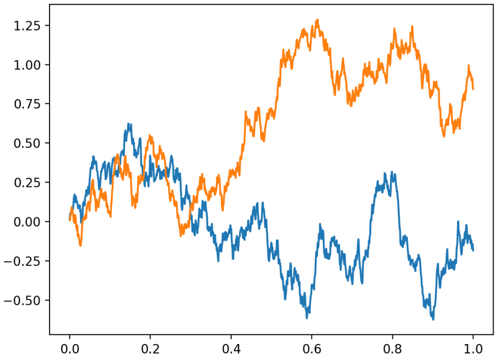
Optimising over correlations between B, W.

Cf. [Bion-Nadal, Talay '19]

Couling SDEs

Example

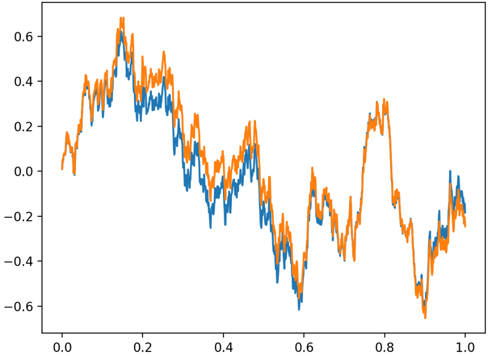
Product coupling — B, W independent



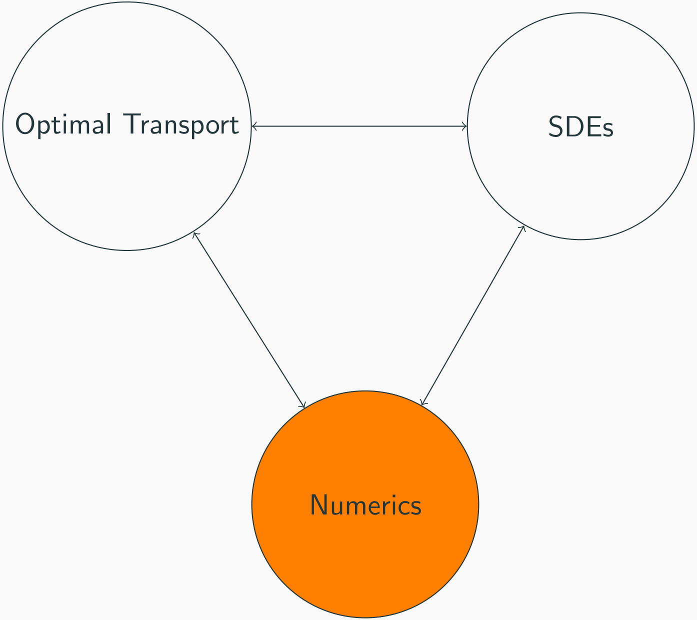
Coupling SDEs

Synchronous coupling

Choose the same driving Brownian motion $B = W$.



Ingredients



Proof of main result

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t,\ X_0 = x \ \rightsquigarrow\ \mathrm{Law}(X) = \mu$$
$$\mathrm{d}\bar{X}_t = \bar{b}(\bar{X}_t)\mathrm{d}t + \bar{\sigma}(\bar{X}_t)\mathrm{d}W_t,\ \bar{X}_0 = x \ \rightsquigarrow\ \mathrm{Law}(\bar{X}) = \nu$$

Theorem [R., Szölgyenyi '24+]

Under "weak assumptions" on the coefficients, we can compute the adapted Wasserstein distance by

$$\mathcal{AW}_2^2(\mu,\nu) = \mathbb{E}\left[\int_0^T |X_t - \bar{X}_t|^2 \mathrm{d}t\right], \quad \text{with } B = W.$$

1. Discretise SDEs;
2. Solve discrete-time bicausal optimal transport problem;
3. Pass to a limit.

Key result in discrete time

$$\mu, \nu \in \mathcal{P}(\mathbb{R}^N) \quad \rightsquigarrow \quad \mathcal{AW}_2^2(\mu, \nu) := \inf_{\substack{T:\, T_\# \mu = \nu \\ \text{biadapted}}} \mathbb{E}\left[\sum_{n=1}^N |T_n(X) - X_n|^2\right].$$

Theorem [Rüschendorf '85]

For μ, ν stochastically co-monotone, the unique optimiser is the Knothe–Rosenblatt rearrangement.

Theorem [Backhoff-Veraguas, Källblad, R. '22]

In the case of Lipschitz coefficients, the Euler–Maruyama scheme is stochastically increasing when the Brownian increments are truncated.

Cf. [Milstein, Repin, Tretyakov '02]

Transformation-based semi-implicit Euler scheme

Assumption (A)

Drift $b\colon \mathbb{R} \to \mathbb{R}$ satisfies the following conditions piecewise:

- absolute continuity
- one-sided Lipschitz condition
- two-sided local Lipschitz condition
- exponential growth

Diffusion $\sigma\colon \mathbb{R} \to [0,\infty)$ satisfies

- global Lipschitz condition
- $\sigma(\xi_k) \neq 0$, for $k \in \{1,\ldots,m\}$ — no uniform ellipticity

Transformation-based semi-implicit Euler scheme

Under assumption (A) , the scheme is constructed as follows:

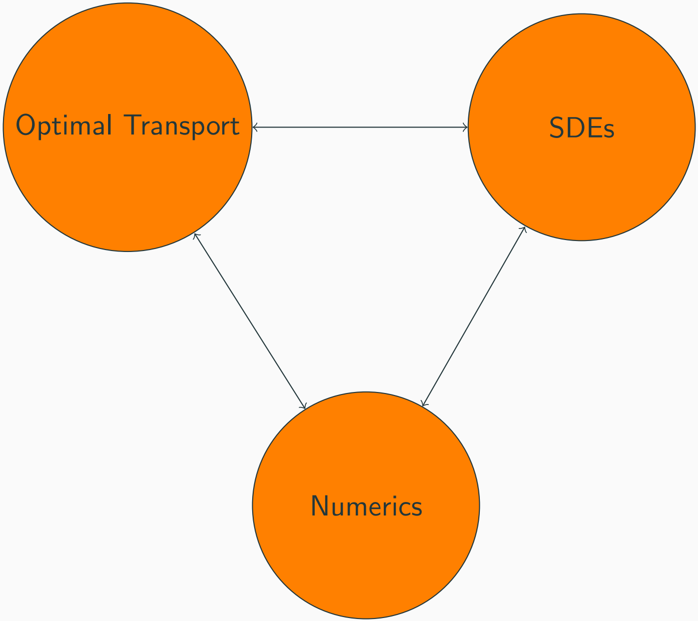
1. Apply the transformation from [Leobacher, Szölgyenyi '17] to (SDE),
2. Apply a semi-implicit Euler scheme with truncated Brownian increments to the transformed SDE, [Hu '96][Higham, Mao, Stuart '02]
3. Transform back.

Denoting by X^h the scheme with step-size h, we obtain the result:

Theorem [R., Szölgyenyi '24]
Let (b, σ) satisfy Assumption 1. Then for all $p \geq 1$, there exists $C_p \geq 0$ such that

$$\mathbb{E}\Big[|X_T - X_T^h|^p\Big]^{\frac{1}{p}} \leq \begin{cases} C_p h^{\frac{1}{2}}, & p \in [1,2], \\ C_p h^{\frac{1}{p(p-1)}}, & p \geq 2. \end{cases}$$

Ingredients



Main result

Assumptions
(A) discontinuous drift with exponential growth;
(B) bounded measurable drift, α-Hölder and uniformly elliptic σ;
(C) "regular" coefficients.

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t,\ X_0 = x \ \rightsquigarrow\ \mathrm{Law}(X) = \mu$$
$$\mathrm{d}\bar{X}_t = \bar{b}(\bar{X}_t)\mathrm{d}t + \bar{\sigma}(\bar{X}_t)\mathrm{d}W_t,\ \bar{X}_0 = x \ \rightsquigarrow\ \mathrm{Law}(\bar{X}) = \nu$$

Main Theorem [R., Szölgyenyi '24+]

Let (b, σ) and $(\bar{b}, \bar{\sigma})$ each satisfy one of assumptions (A), (B), (C). Then, for $p \in [1, \infty)$, the adapted Wasserstein distance is given by

$$\mathcal{AW}_p^p(\mu, \nu) = \mathbb{E}\Big[\int_0^T |X_t - \bar{X}_t|^p \mathrm{d}t\Big], \text{ with } B = W$$

Synchronous coupling solves general bicausal transport problem

Summary

- We compute adapted Wasserstein distance between SDEs;
- We prove strong convergence rates for a new numerical scheme