A regularized Kellerer theorem in arbitrary dimension

Benjamin A. Robinson

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September 7, 2023 — 11^{th} Austrian Stochastic Days, Klagenfurt

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Joint work with

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ETH Zürich

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University of Vienna

Motivating problem

Given probability measures μ, ν on $\mathbb{R}^d$ do there exist random variables M_0, M_1 such that

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For any convex function $v : \mathbb{R}^d \to \mathbb{R}$,

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$$\leq \mathbb{E}[\mathbb{E}[v(M_1)|M_0]] \quad \text{Jensen}$$

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Necessary condition: $\mu \preceq \nu$ **in convex order**

For any convex function $v : \mathbb{R}^d \to \mathbb{R}$,

$$\int v \mathrm{d}\mu \leq \int v \mathrm{d}\nu.$$

... also sufficient [Strassen '68]

Problem statement

Given a family of probability measures $(\mu_t)_{t\in I}$ on $\mathbb{R}^d$, does there exist a mimicking martingale M such that

$$\mathrm{Law}(M_t) = \mu_t, \quad \forall t \in I?$$

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Peacocks

Assume that μ is a peacock; i.e. for any convex function $v : \mathbb{R}^d \to \mathbb{R}$,

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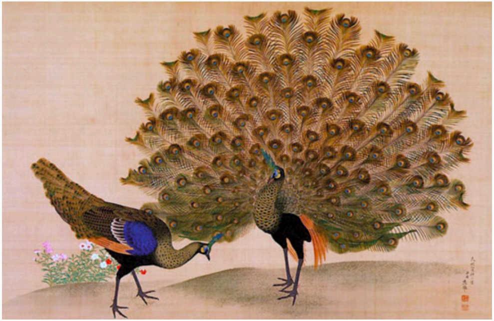
Processus Croissant pour l'Ordre Convexe



[Hirsch, Profetta, Roynette, Yor '11]

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Yes – [Strassen '65, Doob '68, Hirsch–Roynette '13]

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- continuity of paths
- uniqueness

Discrete time

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Given a discrete-time peacock $(\mu_n)_{n\in N}$ on $\mathbb{R}^d$, there exists a mimicking Markov martingale M.

Continuous time, $d = 1$

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Subsequent contributions (incomplete!)

Albin, Baker, Beiglböck, Brückerhoff, Boubel, Donati-Martin, Hamza, Hirsch, Huesmann, Juillet, Källblad, Klebaner, Lowther, Profetta, Roynette, Stebegg, Tan, Touzi, Yor, ...

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- M has the strong Markov property;

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Suppose $t \mapsto \mu_t$ is weakly continuous with convex support. Then

- M has the strong Markov property;
- M is the unique strong Markov mimicking martingale;

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Doob '68 (compact support)

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Existing literature ...

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Doob '68 (compact support), Hirsch–Roynette '13 ($\mathbb{R}^d$)

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no known results

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Theorem 1 [Pammer, R., Schachermayer '22]

There exists a strong Markov martingale diffusion mimicking a *regularized* continuous-time peacock on $\mathbb{R}^d$, $d \in \mathbb{N}$.

Continuous time, $d \geq 2$

Weakly continuous $\mathbb{R}^d$-valued square-integrable peacock $(\mu_t)_{t\in[0,1]}$.
Regularize with a Gaussian $\mu_t^{\mathrm{r}} := \mu_t * \gamma^{\varepsilon(t+\delta)}$

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Proof idea

- Discretise and take Bass martingales from μ_{t_k} to $\mu_{t_{k+1}}$ to get a diffusion process
  [Backhoff, Beiglböck, Huesmann, Källblad '19]
  [Backhoff, Beiglböck, Schachermayer, Tschiderer '23]

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- Regularize with a Gaussian and make a Markovian projection [Krylov '85], [Gyöngy '85], [Brunick, Shreve '13]

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$$\mathrm{d}X_t = \sigma_t \mathrm{d}W_t$$

$$\mathrm{d}\hat{X}_t = \hat{\sigma}_t(\hat{X}_t)\mathrm{d}W_t, \quad \mathrm{Law}(X_t) = \mathrm{Law}(\hat{X}_t),\ t \in [0,1]$$

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$$\mathrm{Law}(\hat{X}_{t_k}) = \mu^{\mathrm{r}}_{t_k} \quad \text{and} \quad \hat{\sigma} \text{ “nice”}$$

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- Use stability of diffusions to pass to a limit

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  For $\mathrm{d}X_t^k = \sigma_t^k(X_t^k)\mathrm{d}B_t$ for "nice" σ^k, suppose for each (t, x)

$$\int_0^t \sigma_s^k(x)^2 \mathrm{d}s \to \int_0^t \sigma_s(x)^2 \mathrm{d}s.$$

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  Then $X^k \to X$ in f.d.d., $\mathrm{d}X_t = \sigma_t(X_t)\mathrm{d}B_t$ and σ "nice".

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Faking Brownian motion

Do there exist stochastic processes with Brownian marginals that are not Brownian motion?

[Hamza, Klebaner '07]

There exists some fake Brownian motion.

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There exists a "very fake" Brownian motion in dimension $d = 1$.

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There exists a Markov process with continuous paths that mimics Brownian marginals in dimension $d = 1$.

Non-uniqueness

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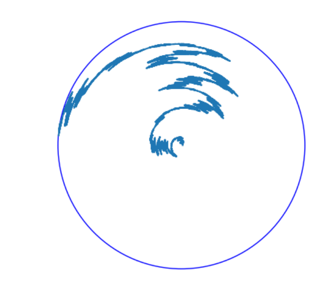
Circular Brownian Motion
[Émery, Schachermayer '99]
[Fernholz, Karatzas, Ruf '18]
[Larsson, Ruf '20]



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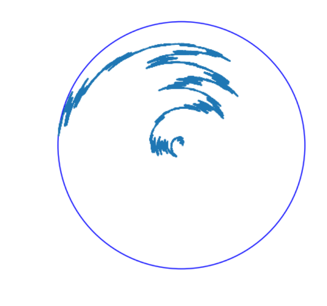
Theorem [Cox, R. '22]
There is a unique weak solution but no strong solution of

$$\mathrm{d}X_t = \frac{1}{|X_t|}\begin{bmatrix} -X_t^2 \\ X_t^1 \end{bmatrix} \mathrm{d}W_t, \quad X_0 = 0.$$



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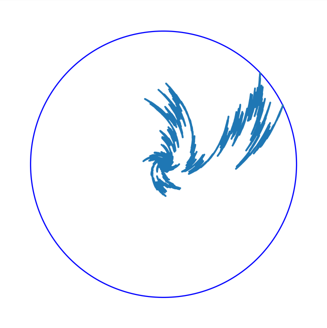
Theorem [Cox, R. '22]
Let X be a weak solution of

$$\mathrm{d}X_t = \frac{1}{|X_t|}(X_t + X_t^{\perp})\mathrm{d}W_t,\ X_0 \sim \eta.$$

Then X is a continuous strong Markov fake Brownian motion.



[Pammer, R., Schachermayer '22]

Counterexamples

Theorem 4 [Pammer, R., Schachermayer '22]

There exists a weakly continuous square-integrable peacock $(\mu_t)_{t\in[0,1]}$ on $\mathbb{R}^4$ such that, for the peacock $(\mu_t * \gamma^t)_{t\in[0,1]}$, there exists no mimicking Markov martingale.

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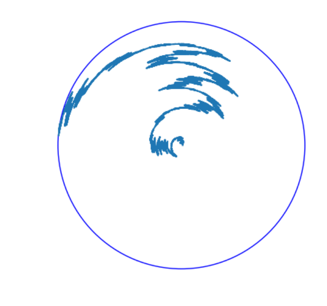

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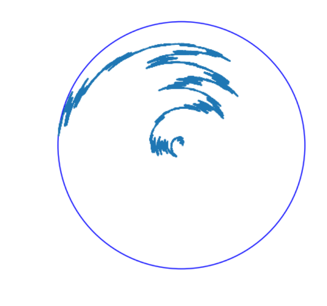

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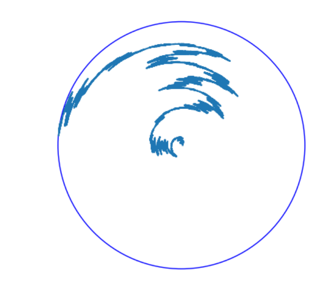
1. No continuous Markov martingale mimicking μ;
2. No Markov martingale mimicking μ;



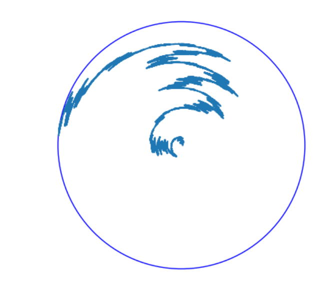
[Cox, R. '22]

Counterexamples

Theorem 4 [Pammer, R., Schachermayer '22]

There exists a weakly continuous square-integrable peacock $(\mu_t)_{t\in[0,1]}$ on $\mathbb{R}^4$ such that, for the peacock $(\mu_t * \gamma^t)_{t\in[0,1]}$, there exists no mimicking Markov martingale.

1. No continuous Markov martingale mimicking μ;
2. No Markov martingale mimicking μ;
3. No Markov martingale mimicking $(\mu * \gamma^t)_{t\in[0,1]}$.



[Cox, R. '22]

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Summary

- We prove the first known Kellerer-type result in arbitrary dimension;
- In dimension $d \geq 2$, uniqueness fails;
- In general, the result can fail without some regularization.

arXiv:2210.13847